Fall 2015

1. 10 apples for \$6 is \$0.60 per apple. 20 apples for \$10 is \$0.50 per apple.

$25 * \$0.60 - 25 * \$0.50 = 25 * \$0.10 = \$2.50$

So the answer is $\mathbf{B. \$2.50}$

2. Let's begin by finding the slopes of the two lines.

$ax + 12y = 6$
$12y = -ax + 6$
$y = -\frac{a}{12}x + \frac{1}{2}$
$m_1 = -\frac{a}{12}$
$ax - 3y = 12$
$-3y = -ax + 12$
$y = \frac{a}{3}x - 4$
$m_2 = \frac{a}{3}$
Perpendicular means that the slopes are negative reciprocals, so
$\frac{12}{a} = \frac{a}{3}$
$36 = a^2$
$6 = a$ (since the problem stipulated that a is nonnegative).

So the answer is $\mathbf{D. 6}$

3. b, c, and d are all integers and not equal to each other.

$\frac{1}{1} +$ anything else is larger than 1 so b, c, and d are all greater than 1.
$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} < 1$, so b must be less than 3, thus b is 2.
$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < 1$, so c must be less than 4, thus c is 3.
$1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so d is 6.
So $b + c + d = 2 + 3 + 6 = 11$

So the answer is $\mathbf{C. 11}$

4. 4 and 5 both divide the number of students equally, so 20 does as well. The number also has to be 2 less than a multiple of 6, so it is 40 + a multiple of 60. Since it is an 8th grade math class, we can assume it is less than 100, so it is 40. Adding 4 students gets us 44, which is divisible by 11.

So the answer is $\mathbf{E. 11}$

5. Since $x = 12$ is a solution of $2x + p = q, 24 + p = q$.

Substituting into $3x + q = p$ gets us $3x + 24 + p = p$
$3x + 24 = 0$
$3x = -24$
$x = -8$

So the answer is **B. -8**

6. 1 of each coin is 4 coins worth \$.41, leaving 38 coins worth \$.59. Adding a quarter would leave 37 coins worth \$.34, which is impossible. Adding a dime would leave 37 coins worth \$.49. Another dime would leave 36 coins and \$.39, which is impossible since a nickel would leave 35 coins worth \$.34. So we add 3 nickels instead, leaving 34 coins worth \$.34, which works. So to the initial one of each coin we have added 1 dime and 3 nickels, for a total of 2 dimes, 4 nickels, 6 together.

So the answer is **D. 6**

7. P(x) is a third degree polynomial so $P(x) = ax^3 + bx^2 + cx + d$.

From what we are given we can get 4 equations:
$-a + b - c + d = 2$
$a + b + c + d = 2$
$8a + 4b + 2c + d = 2$
$27a + 9b + 3c + d = 10$

We can use our TI 83 or TI 84 to solve this by putting it into a matrix and using the rref command on it (from the matrix menu), giving us $(a, b, c, d) = (1, -2, -1, 4)$. So $P(x) = x^3 - 2x^2 - x + 4$
Plugging in 4 gets us 32.

So the answer is **E. 32**

8. The side perpendicular to the face with diagonal 15 (I'll call it a) forms a right triangle with that diagonal and the long diagonal. So

$a^2 + 15^2 = 17^2$
$a^2 = 64$
$a = 8$
This means that the side of length 9 must be one of the sides making the face with diagonal 15. So the other side (I'll call it b) can also be found:
$b^2 + 9^2 = 15^2$
$b^2 = 144$
$b = 12$
Now we know all the sides so we can just multiply. $9 * 12 * 8 = 864$

So the answer is **E. 864**

9. Let's use A for the price of the first, B for the second, C for the third. Let's assume C is 2. Then $B \geq 3$. So $A > 6$ and $A < 6$. That doesn't work. If C is 3, $B \geq 4$, $8 < A < 9$, which isn't possible since A needs to be an integer. If C is 4, $B \geq 5$, $10 < A < 12$, so B=5, A=11. $11 + 5 + 4 = 20 \neq 25$, so that doesn't work. If C is 5, B is 6 since if B was 7, A couldn't be an integer. $25 - 6 - 5 = 14$, so A is 14. $12 < 14 < 15$, so that works. Therefore C is 5.

So the answer is **D. 5**

10. Our two points are $(a, a^2)$, and $(-b, b^2)$. Let's start by finding the slope.

$m = \frac{a^2 - b^2}{a+b} = \frac{(a-b)(a+b)}{a+b} = a - b$
Plugging this and the first point into the slope point formula:
$y - a^2 = (a-b)(x-a)$
$y = (a-b)x - a(a-b) + a^2$
$y = (a-b)x - a^2 + ab + a^2$
$y = (a-b)x + ab$
This is a line with a slope of $a - b$ and an intercept of $ab$

So the answer is **A. ab**

11. Let's just look at integer values we can have for right triangles. 3k, 4k, 5k or 5k, 12k, 13k are the smallest two. In a 3k, 4k, 5k triangle the perimeter is 12k and the area is $6k^2$, so they are equal when k is 2 with area = perimeter = 24. In a 5k, 12k, 13k, the perimeter is 30k and the area is $30k^2$, which are equal when k is 1, for a perimeter and area of 30. The next integer right triangle is 7k, 24k, 25k, which has a perimeter of 56k and an area of $84k^2$. This gives no positive integer values of k where they are equal, and this holds fo r any larger right triangle with integer sides. So 30 is the largest we can get.

So the answer is **C. 30**

12. Using a TI 83 or TI 84, we put into Y= √4X+9 and √9X+1, then look at the table. At 0 we have 3 and 1, and at 40 we have 13 and 19. 3+1+13+19=36

So the answer is **D. 36**

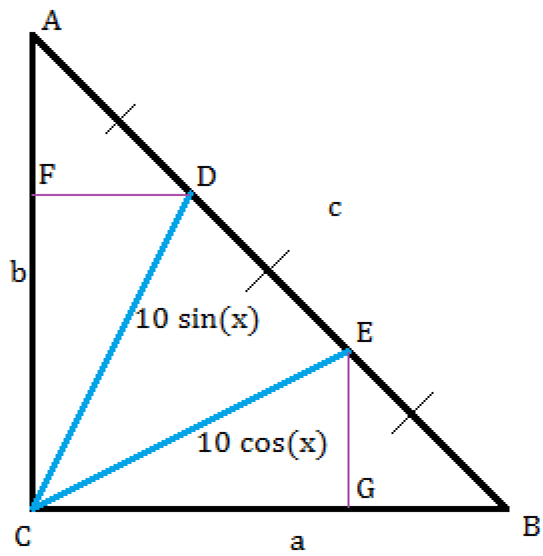
13. In order for a knight to be telling the truth, he must be sitting next to one knight and one knave. For the knave sitting next to a knight to be lying, there must be a knight on either side. This means there must be a pattern of knigh t, knight, knave, knight, knight, knave, etc. So the number of knaves is 1/3 of the total. Since the total is 12N, this is 4N.

So the answer is **C. 4N**

14. Let's start by looking at the ones digit. There are 5! ways it can be a 1, and due to the repeated 1, there are 5!/2 ways it can be each of the other numbers. so the sum of all the ones digits will be 5!(1)+5!/2(2+3+5+8)=5!(10). Since the same will hold for all the digits, our answer will be 5!(10)(111111)=133333200.

So the answer is **B. 133333200**

15.




We are given all of the information in this image except for the perpendiculars at F and G and need to find side c. Note that since AFD is similar to ACB, AF is 1/3 of side b, and FC is 2/3 of b. Similarly, CG is 2/3 of a and BG is 1/3 of a. Since CGE and CFD are right triangles:

$$\left(\frac{2}{3}b\right)^2 + \left(\frac{1}{3}a\right)^2 = (10\sin x)^2$$

$$\left(\frac{1}{3}b\right)^2 + \left(\frac{2}{3}a\right)^2 = (10\cos x)^2$$

$$\frac{5}{9}b^2 + \frac{5}{9}a^2 = 100\cos^2 x + 100\sin^2 x$$

$$\frac{5}{9}(b^2 + a^2) = 100$$

$$c^2 = \frac{900}{5} = 180$$

$$c = \sqrt{180} = 6\sqrt{5}$$

So the answer is **A. $6\sqrt{5}$**

16. The only way to get a value of \$38 with the given cards is 2 each of the \$13, \$5, and \$1 cards, leaving 2 of each in the deck. So to get a value of 38 or more, the next 6 cards would need to be the 6 remaining cards of value. There are ${}_{40}C_6$ possible hands and that is only 1 of them so our probability is $1/{}_{40}C_6 \approx$ .00000026

So the answer is **A. 0.00000026**

17. $a^5 + b^3 + c^2 = 2015$

$c = \sqrt{2015 - b^3 - a^5}$

From here we'll let our calculator do the work. My instructions assume a TI-83 or TI-84 is used. Go into Y= and type in √(2015 –X^3–1^5), then look at the table. We don't get any integer solutions here, so go back to Y= and replace the 1 with a 2. Still no integer results so try a 3. Now we get 2 results: 2 gives 42, and 11 gives 21. These correspond to $(a, b, c) = (3, 2, 42)$ and $(3, 11, 21)$. $3 * 2 = 6$ is a factor of 42, so this is the solution we want and $\frac{42}{6} = 7$

So the answer is **E. 7**

18. There are 2500 possible combinations, so we can figure out the total payoff of every combination and divide by 2500. There are 49 ways for C-A to be 1, 48 ways to get 2, 47 ways to get 3, and so on until finally there is 1 way to get 49. So the total payoff is $1(49) + 2(48) + 3(47) + \cdots + 49(1) =$

$$\sum_{i=1}^{49} i(50 - i) = \sum_{i=1}^{49} 50i - i^2$$

$= \frac{50(49)(49+1)}{2} - \frac{49(49+1)(2(49)+1)}{6} = 20825$

Dividing by 2500 gets us 8.33

So the answer is **D. 8.33**

19. We are given the values of four dice rolls and can set them up as 4 equations:

$Y_1 + B_1 + R_1 = 13$

$Y_2 + B_1 + R_2 = 6$

$Y_2 + B_2 + R_3 = 15$

$Y_3 + B_3 + R_3 = 7$

Since these are 6-sided dice, we know the value of each variable is an integer between 1 and 6. Looking at the second equation, we know that $Y_2, B_1$, and $R_2$ are each at most 4. Looking at the third equation, we see that since $Y_2$ is 4 or less, $B_2$ and $R_3$ are at least 5. The fourth equation tells us $R_3$ is at most 5, so $R_3$ is 5, and $B_3$ and $Y_3$ are each 1. With $Y_2$ at most 4, and $R_3$ as 5, it must be that $Y_2$ is 4 and $B_2$ is 6. This means that $B_1$ and $R_2$ are both 1 (from the second equation), and the first equation then tells us that $Y_1$ and $R_1$ are both 6. So $Y_1 + Y_2 + Y_3 = 6 + 4 + 1 = 11$

So the answer is **D. 11**

20. Let's say $\log_4 m = \log_6 n = \log_9(m+n) = x$.

Then $m = 4^x, n = 6^x,$ and $m + n = 9^x$, so $9^x = 4^x + 6^x$

Dividing by $6^x$ gets us $\left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^x + 1$, with $\frac{m}{n} = \left(\frac{2}{3}\right)^x$ being what we need to find. If we substitute that with $y$, that makes $\left(\frac{3}{2}\right)^x = \frac{1}{y}$, so $\frac{1}{y} = y + 1$

$1 = y^2 + y$

$0 = y^2 + y - 1$

$y = \frac{-1 \pm \sqrt{5}}{2}$

So a=-1, b=5, c=2, and $a + b + c = 6$

So the answer is **A. 6**